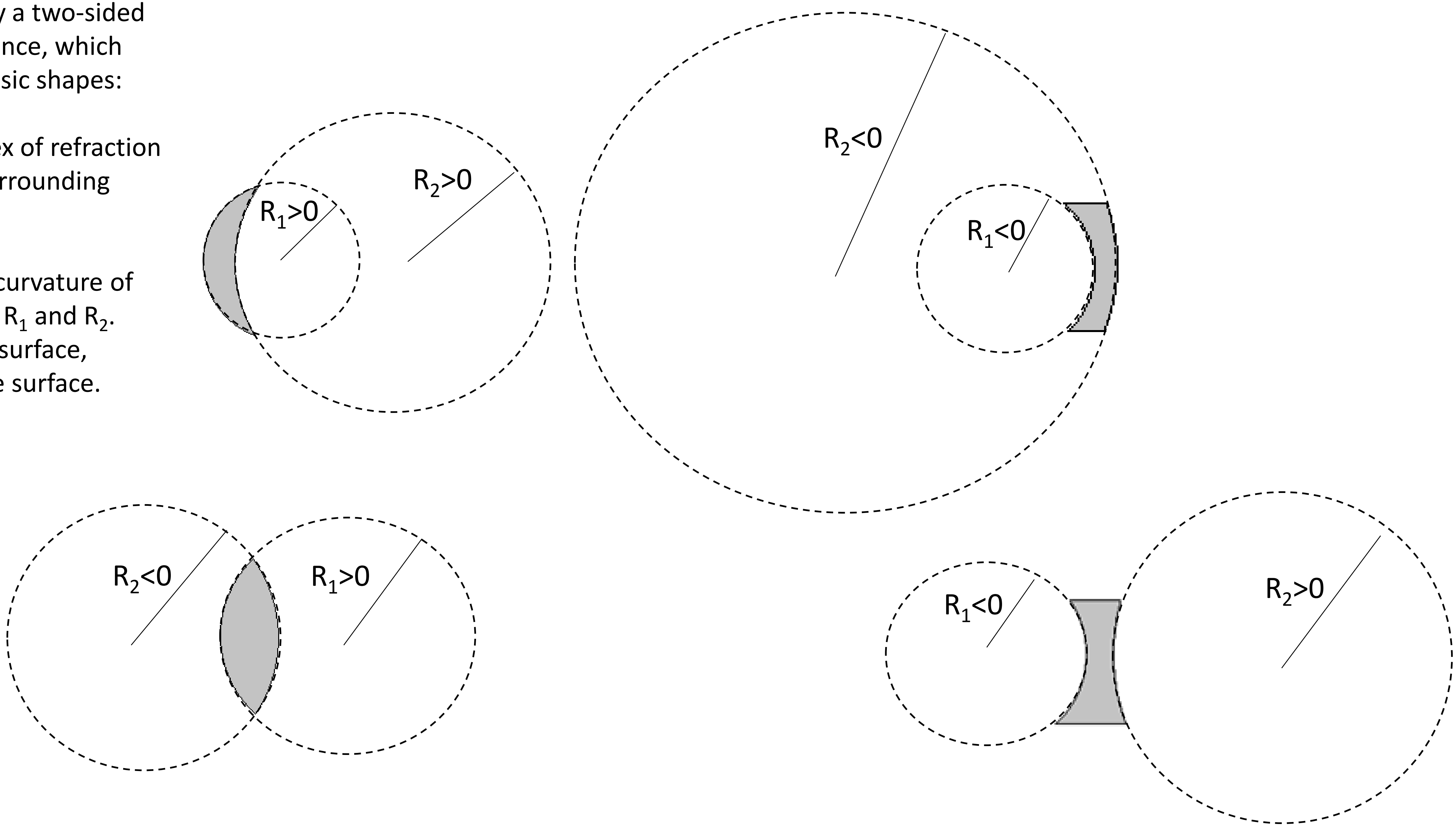
E.4 Lenses

A lens is basically a two-sided refracting substance, which comes in four basic shapes:

We'll call its index of refraction n_2, and of the surrounding medium n_1.

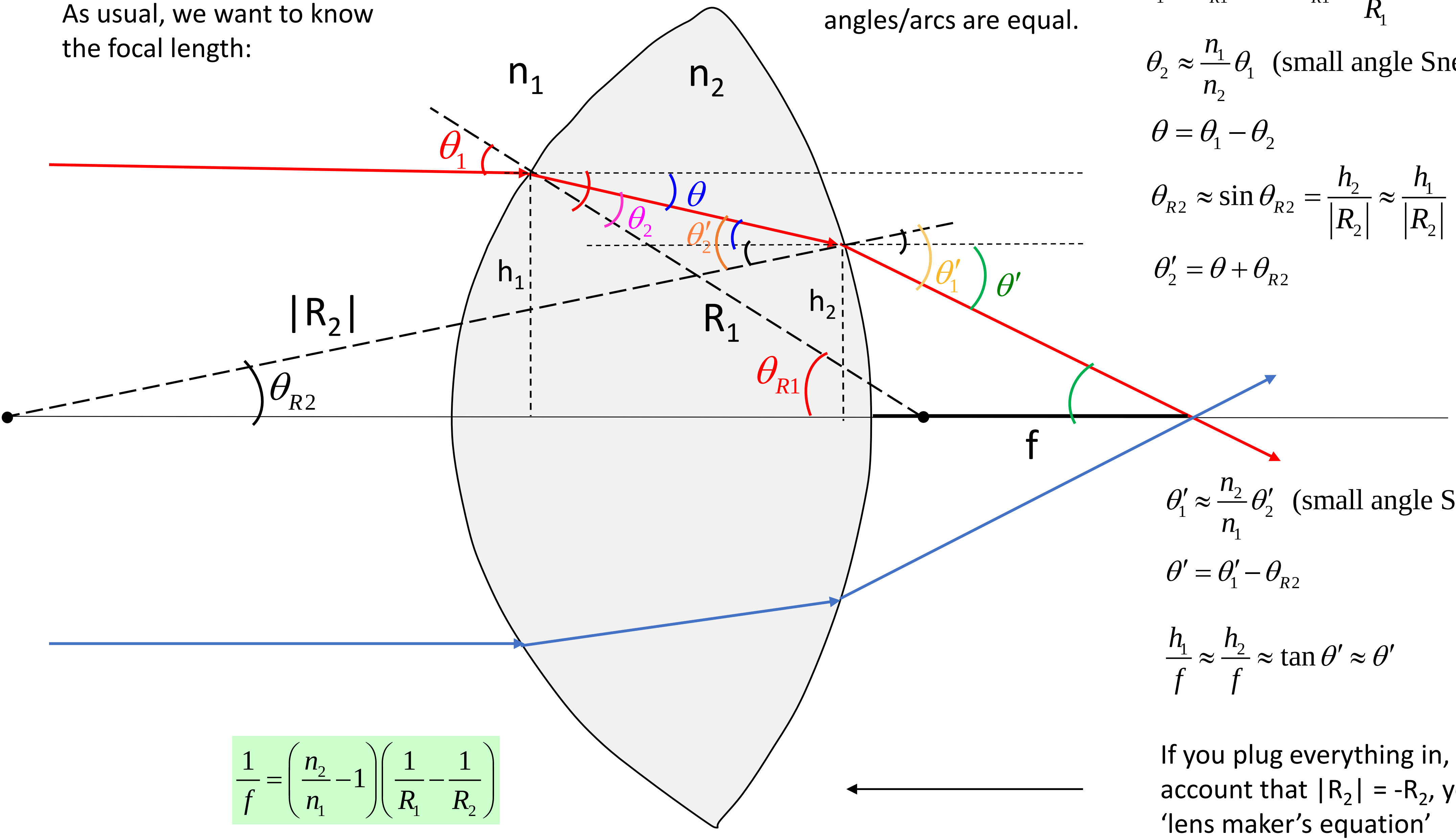
And the radii of curvature of the sides we call R_1 and R_2.
R > 0 for convex surface,
R < 0 for concave surface.



E.4 Lenses

As usual, we want to know the focal length:

Note that same-colored angles/arcs are equal.



$$\theta_1 = \theta_{R1} \approx \sin\theta_{R1} = \frac{h_1}{R_1}$$

$$\theta_2 \approx \frac{n_1}{n_2}\theta_1 \quad \text{(small angle Snell's law)}$$

$$\theta = \theta_1 - \theta_2$$

$$\theta_{R2} \approx \sin\theta_{R2} = \frac{h_2}{|R_2|} \approx \frac{h_1}{|R_2|} \quad \text{(if lens is thin)}$$

$$\theta_2' = \theta + \theta_{R2}$$

$$\theta_1' \approx \frac{n_2}{n_1}\theta_2' \quad \text{(small angle Snell's law)}$$

$$\theta' = \theta_1' - \theta_{R2}$$

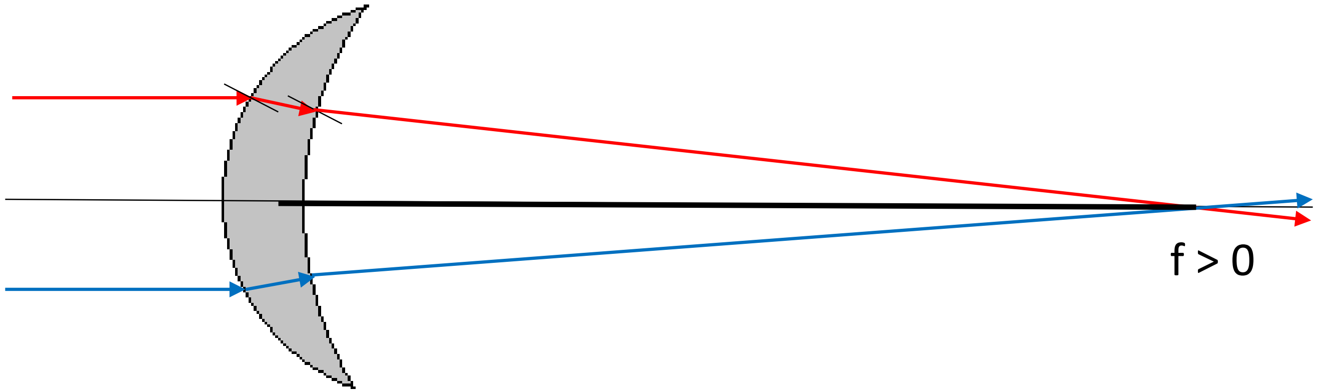
$$\frac{h_1}{f} \approx \frac{h_2}{f} \approx \tan\theta' \approx \theta'$$

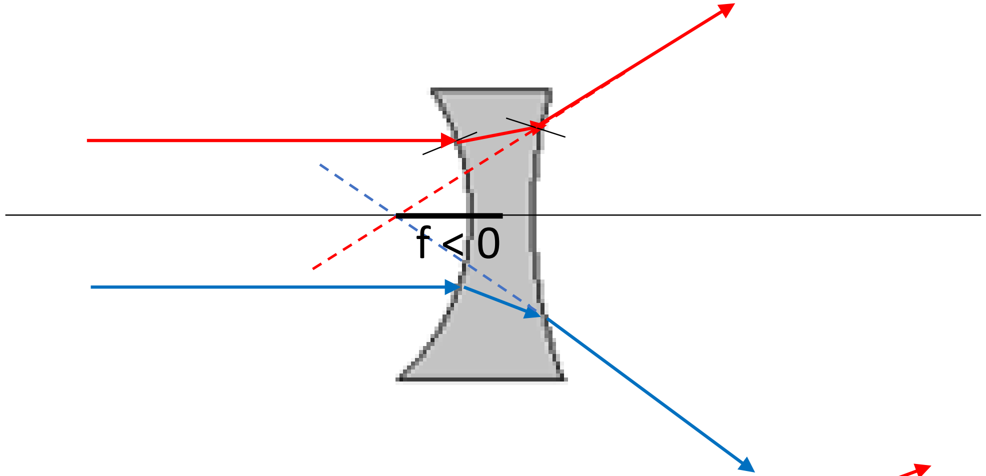
If you plug everything in, from top down, taking account that $|R_2| = -R_2$, you'll get the 'lens maker's equation'

$$\boxed{\frac{1}{f} = \left(\frac{n_2}{n_1} - 1\right)\left(\frac{1}{R_1} - \frac{1}{R_2}\right)}$$

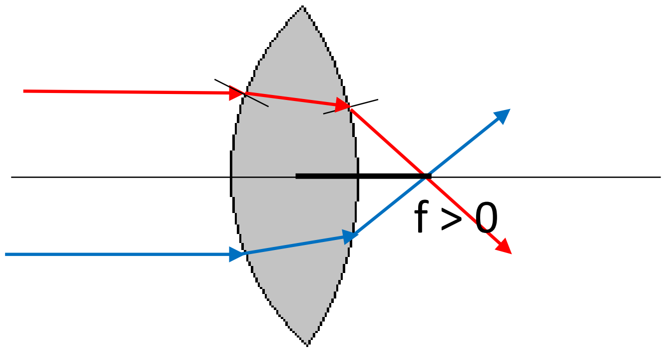
E.4 Lenses

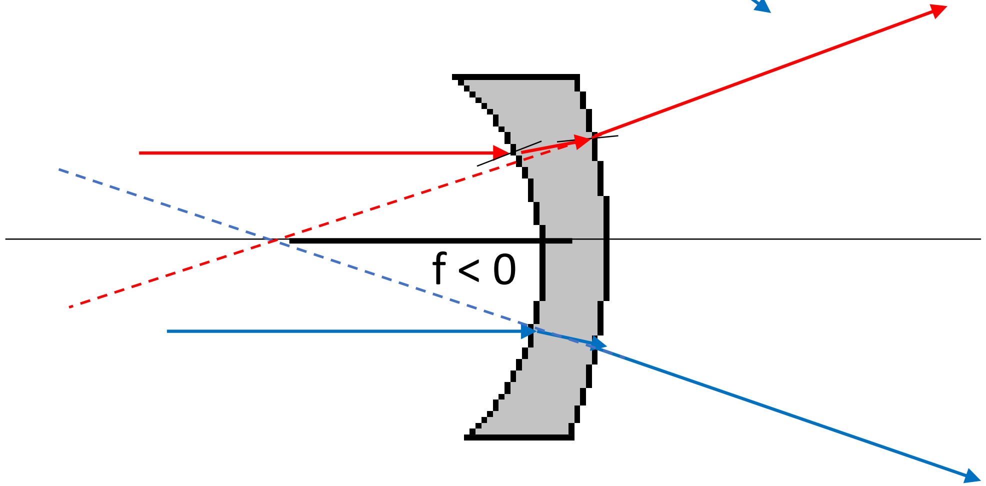
That was fun! Now we only have three more shapes to go. But luckily for us, it turns out that they all obey the same formula…





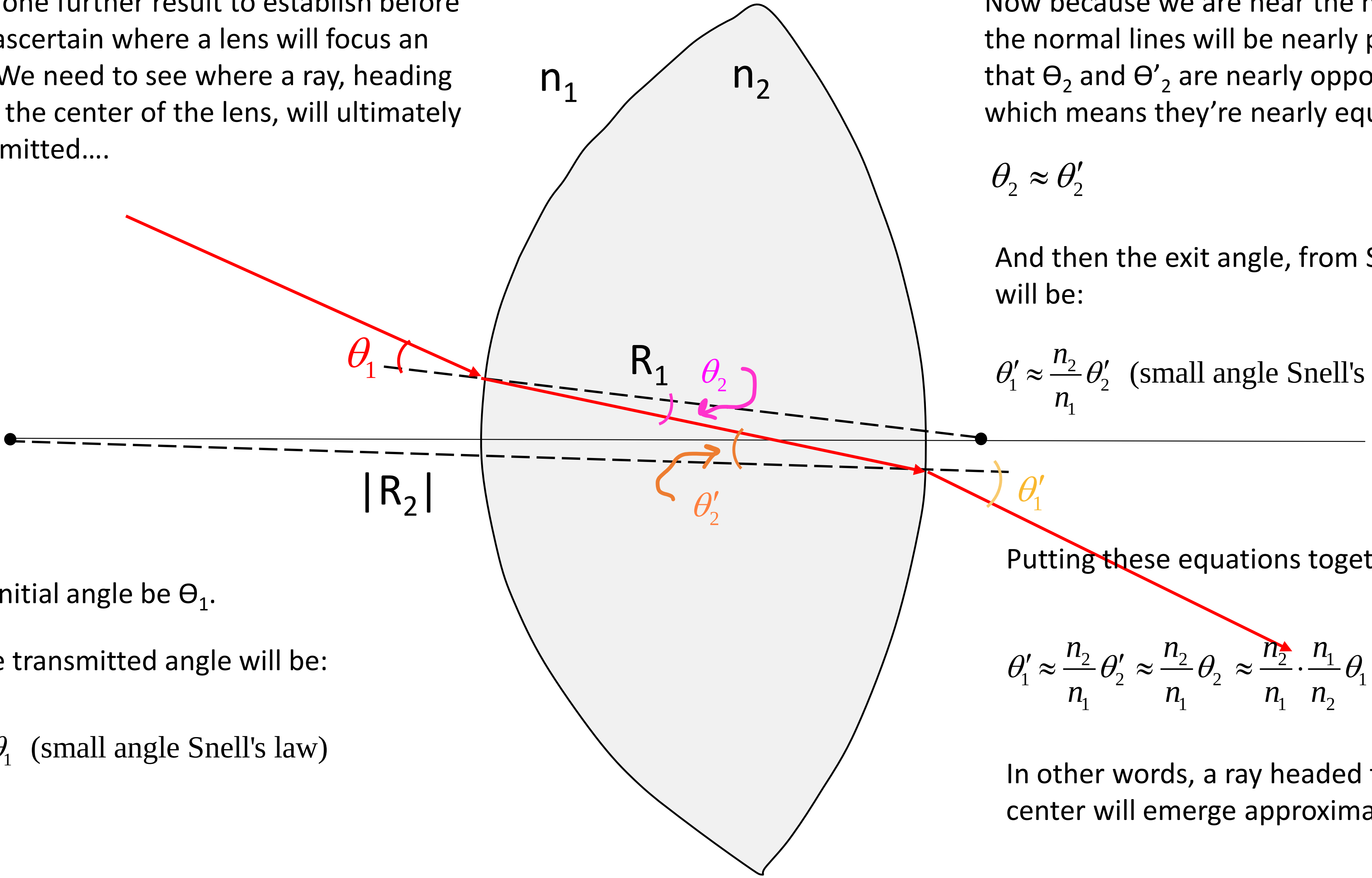
$$\frac{1}{f}=\left(\frac{n_2}{n_1}-1\right)\left(\frac{1}{R_1}-\frac{1}{R_2}\right)$$





E.4 Lenses

There is one further result to establish before we can ascertain where a lens will focus an image. We need to see where a ray, heading towards the center of the lens, will ultimately be transmitted….



Let the initial angle be Θ_1.

Then the transmitted angle will be:

$$\theta_2 \approx \frac{n_1}{n_2}\theta_1 \quad \text{(small angle Snell's law)}$$

Now because we are near the middle of the lens, the normal lines will be nearly parallel. This means that Θ_2 and Θ'_2 are nearly opposite interior angles, which means they're nearly equal:

$$\theta_2 \approx \theta_2'$$

And then the exit angle, from Snell's law again, will be:

$$\theta_1' \approx \frac{n_2}{n_1}\theta_2' \quad \text{(small angle Snell's law)}$$

Putting these equations together, we have:

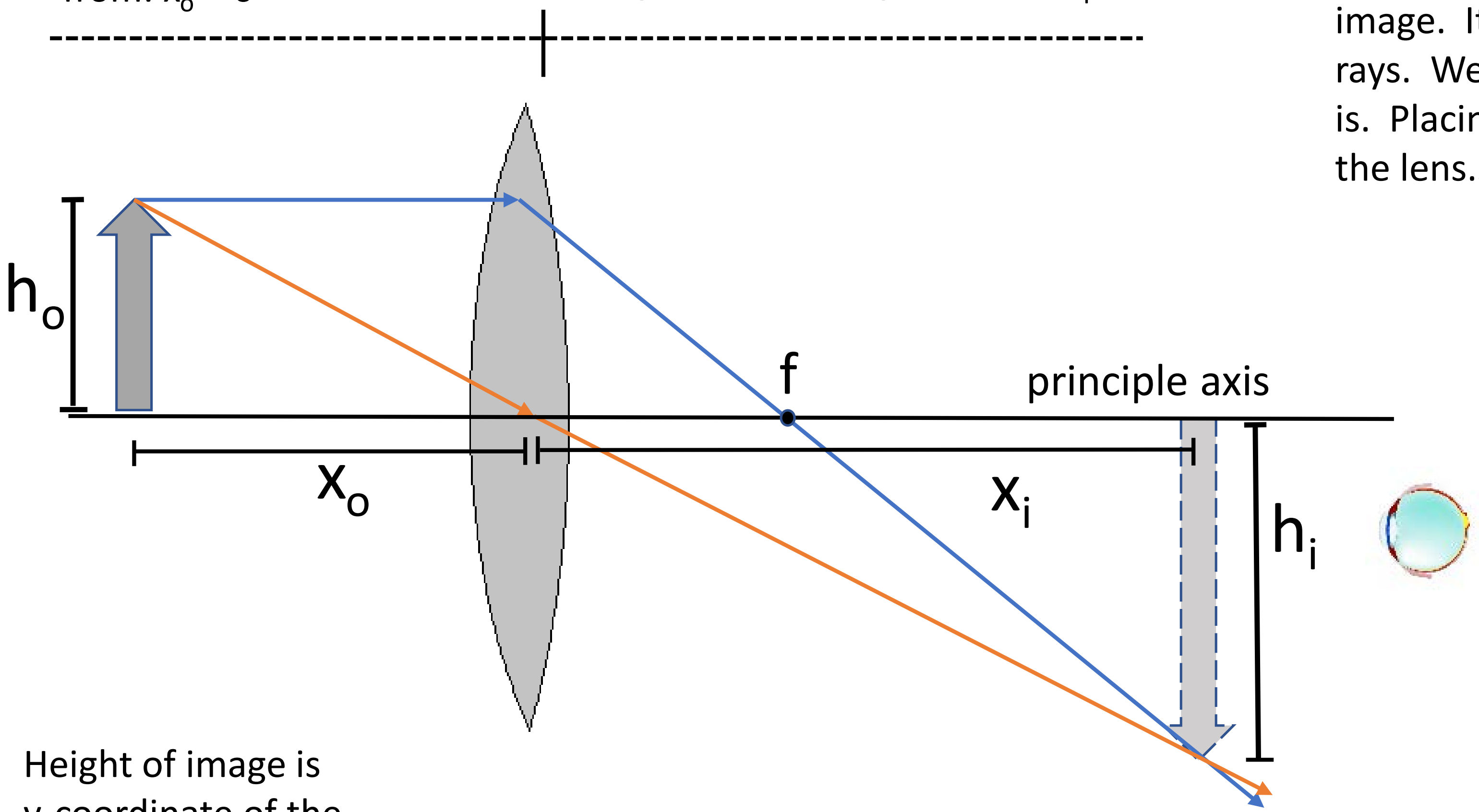
$$\theta_1' \approx \frac{n_2}{n_1}\theta_2' \approx \frac{n_2}{n_1}\theta_2 \approx \frac{n_2}{n_1}\cdot\frac{n_1}{n_2}\theta_1 = \theta_1$$

In other words, a ray headed towards the center will emerge approximately undeflected.

E.4 Lenses

region where light came from: $x_o > 0$

region where light went: $x_i > 0$



Height of image is y-coordinate of the intersection:

$$h_i = -\frac{h_o}{x_o} x_i$$

So magnification is: $$\boxed{m \equiv \frac{h_i}{h_o} = -\frac{x_i}{x_o}}$$

Now we have enough information to locate an object's image. It's located at the intersection of the two reflected rays. We'll write an equation for them and see where this is. Placing an invisible coordinate system at the center of the lens….

Blue transmission:

$$y_{blue} = y_0 + slope \cdot x = h_0 - \frac{h_o}{f} x$$

Orange transmission:

$$y_{orange} = y_0 + slope \cdot x = 0 - \frac{h_o}{x_o} x$$

Intersection:

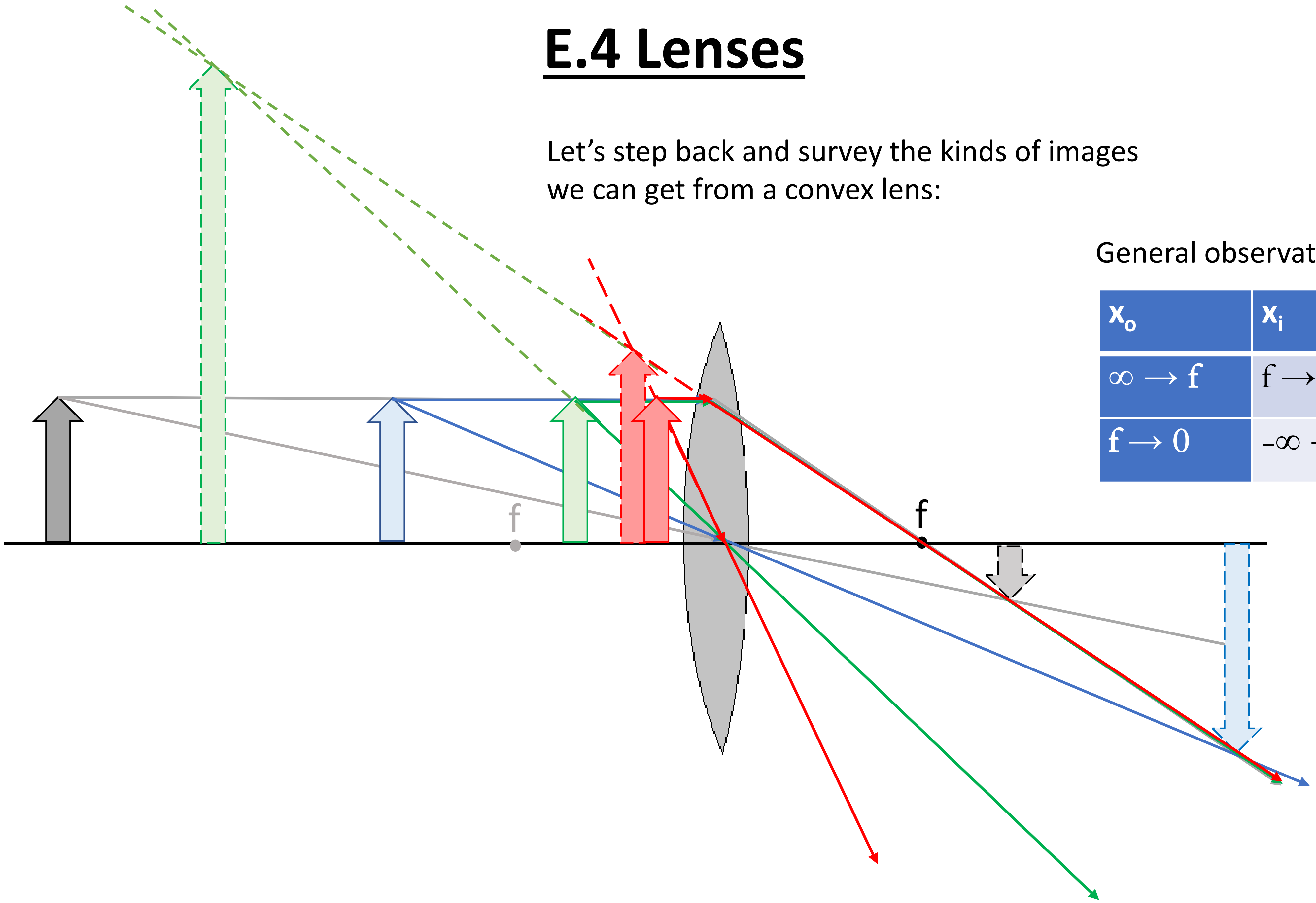
$$y_{blue} = y_{orange} \longrightarrow \cancel{h_0} - \frac{\cancel{h_o}}{f} x_i = -\frac{\cancel{h_o}}{x_o} x_i$$

$$\downarrow$$

$$\boxed{\frac{1}{x_o} + \frac{1}{x_i} = \frac{1}{f}} \longleftarrow 1 - \frac{1}{f} x_i = -\frac{1}{x_o} x_i$$

E.4 Lenses

Let's step back and survey the kinds of images we can get from a convex lens:

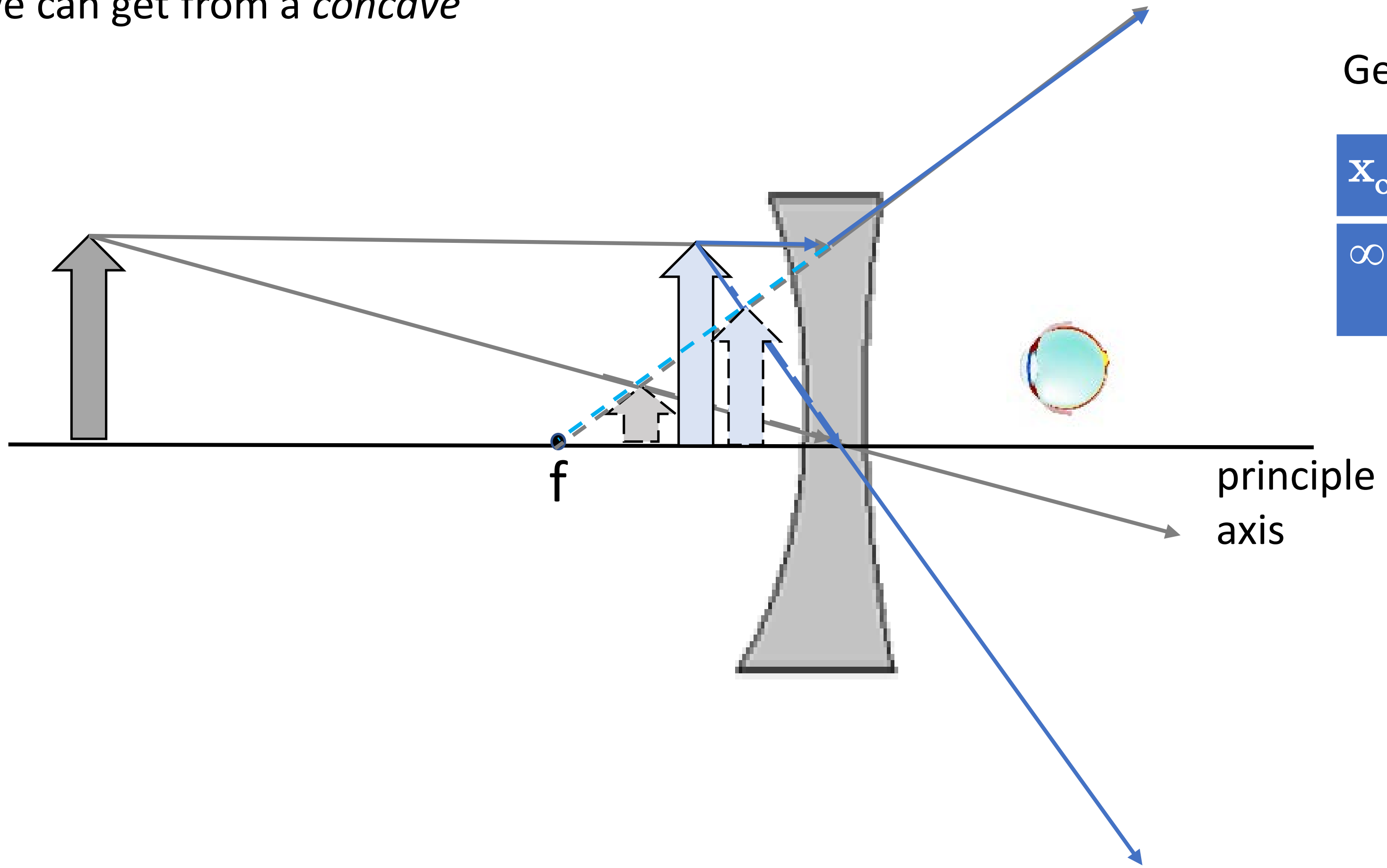


General observations:

x_o	x_i	m	type
$\infty \rightarrow f$	$f \rightarrow \infty$	$0 \rightarrow -\infty$	real
$f \rightarrow 0$	$-\infty \rightarrow 0$	$\infty \rightarrow 1$	virtual

E.4 Lenses

And now let's survey the kinds of images we can get from a *concave* lens:



General observations:

x_o	x_i	m	type
$\infty \rightarrow 0$	$-f \rightarrow 0$	$0 \rightarrow 1$	virtual

E.4 Lenses

Say you need to project a 1cm tall film reel onto a movie screen 20m away, and the image has to be 3m tall. Being an auteur movie house, you also grind your own lenses. So determine what radius of curvature, R, you should grind your symmetric, convex, n = 1.33 acrylic lens to have, and how far from the lens you should place the film.

$$\frac{1}{x_o}+\frac{1}{x_i}=\frac{1}{f}$$

$$\frac{1}{d}+\frac{1}{20}=\frac{1}{f}$$

$$\frac{1}{0.067}+\frac{1}{20}=\frac{1}{f}$$

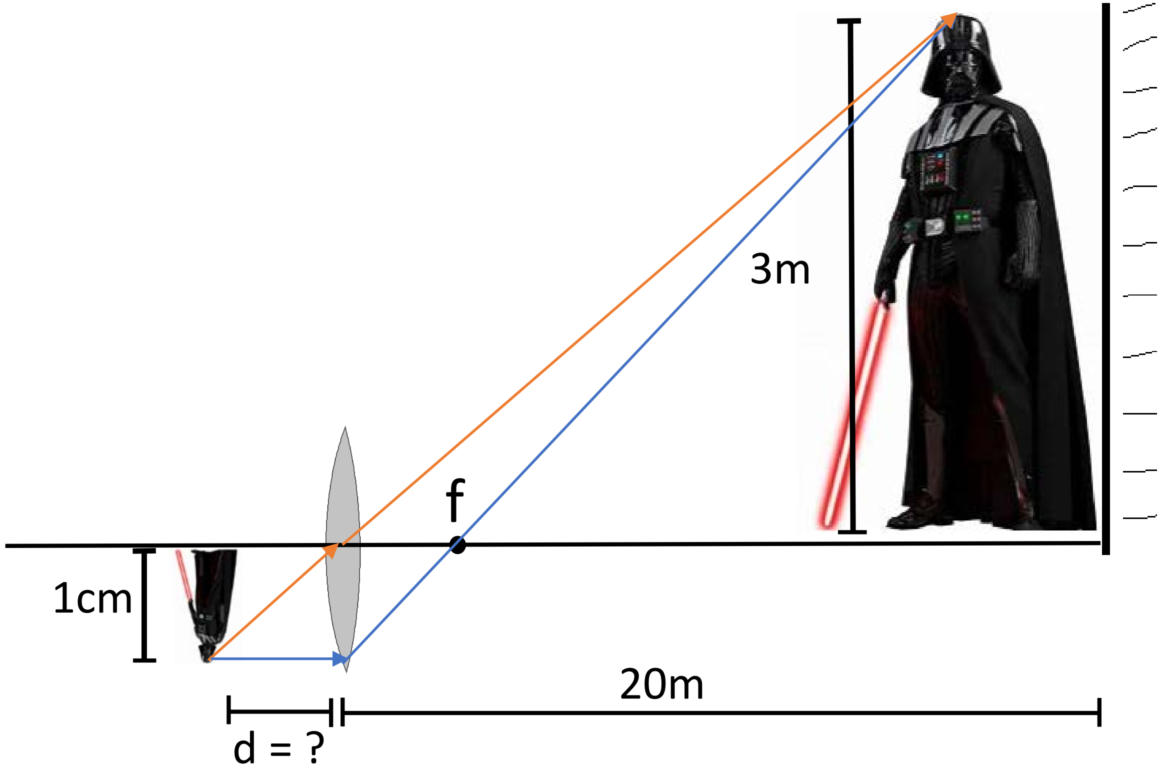
$$f\approx 0.067\text{m}=6.7\text{cm}$$

$$m\equiv\frac{h_i}{h_o}=\frac{3}{-0.01}=-300$$

$$-\frac{x_i}{x_o}=-300$$

$$-\frac{20}{d}=-300$$

$$\boxed{d=-\frac{20}{-300}=0.067\,\text{m}}$$



Then to get the radius of curvature we have to use the lens maker's equation:

$$\frac{1}{f}=\left(\frac{n_2}{n_1}-1\right)\left(\frac{1}{R_1}-\frac{1}{R_2}\right)$$

$$\frac{1}{6.7\text{cm}}=\left(\frac{1.33}{1}-1\right)\left(\frac{1}{R}-\frac{1}{-R}\right)$$

$$\frac{1}{6.7\text{cm}}=(0.33)\left(\frac{2}{R}\right)$$

$$\boxed{R=\frac{2}{0.33}(6.7\text{cm})=40\text{cm}}$$